Half Range series

Theorem :- (The sine series) If 'f' is bounded, integrable and piecewise monotonic in $[0,\pi]$, then the sum of sine series

$$\sum_{n=1}^{\infty} b_n \sin nx \; ; \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx.$$

is equal to $\frac{1}{2}[f(x^+) + f(x^-)]$ at every point x between 0 and π and is equal to 0 at $x = 0, \pi$

Pf:- We know that an odd function $F(x)$ in $[-\pi,\pi]$ will be identified with 'f' in $[0,\pi]$ as,

$$F(x) = \begin{cases} f(x) & 0 \le x \le \pi \\ -F(-x) = -f(-x) & -\pi \le x \le 0 \end{cases}$$

Clearly, $F(x)$ satisfies the condition of the main theorem

The Main Theorem :- Let 'f' be bdd, integrable and piecewise monotonic on $[-\pi,\pi]$. Then

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \begin{cases} \frac{1}{2}[f(x^-) + f(x^+)] \; ; & -\pi < x < \pi \\ \frac{1}{2}[f(\pi^-) + f(-\pi^+)] \; ; & x = \pm\pi \end{cases}$$

$\therefore$ Sum of series

$$\sum_{n=1}^{\infty} b_n \sin nx \; ; \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

is equal to

$$\frac{1}{2}[F(x^+) + F(x^-)]$$

$= \frac{1}{2}[f(x^+) + f(x^-)]$ at every point x between 0 & π

At $x = 0$, sum of series $= \frac{1}{2}[F(0^+) + F(0^-)] = 0$

Similarly at $x = \pi$, sum of series $= 0$. [$\because$ F is odd]

Theorem:- (The cosine Series) If 'f' is bounded, integrable and piecewise monotonic in $[0,\pi]$, then the sum of the cosine series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad ; \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

is equal to $\frac{1}{2}[f(x^-) + f(x^+)]$ at every point 'x' between ~~0~~ and π

$\Big| \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$f(0+)$ at $x=0$, $f(\pi-)$ at $x=\pi$

Pf:- The proof is similar as above after defining an even function F defining in $[-\pi,\pi]$ which is identical with 'f' in $[0,\pi]$

Ex. I Express $\sin x$ as a ~~sine series and~~ cosine series. in $0<x<\pi$

Solⁿ:- We have $f(x) = \sin x$. defined in $0<x<\pi$ may be extended as an even function.

$f(x) = |\sin x|$ in $[-\pi,\pi]$ so that we have.

$$f(x) = \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{———} \quad (*)$$

& $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$, $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = -\frac{2}{\pi}\left[\cos\pi - \cos 0\right]$$

$$\Rightarrow a_0 = -\frac{2}{\pi}[-1-1] = \frac{4}{\pi}$$

$$\therefore \boxed{a_0 = \frac{4}{\pi}}$$

Also $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$.

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{(n+1)} + \frac{\cos(n-1)x}{(n-1)}\right]_0^{\pi} \quad ; n \neq 1$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{(n+1)} + \frac{\cos(n-1)\pi}{(n-1)} + \frac{1}{n+1} - \frac{1}{n-1}\right] ; n \neq 1$$

$$= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] ; n \neq 1$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^{n-1}}{n^2-1} - \frac{2}{n^2-1}\right] ; n \neq 1$$

$$= \frac{2}{\pi(n^2-1)}\left[(-1)^{n-1} - 1\right] ; n \neq 1$$

$$a_n = \begin{cases} 0 & ; n = 3, 5, 7, \ldots \\ \dfrac{-4}{(n^2-1)(\pi)} & ; n \text{ is even.} \end{cases}$$

When $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\,dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$\boxed{a_1 = 0}$$

Substituting these values in ⊛, we get

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{1.3} + \frac{\cos 4x}{3.5} + \frac{\cos 6x}{5.7} + \cdots\right]$$

Also $f(x) = \sin x$ in $(-\pi, \pi)$ may be extended to an odd function. so that

$$\sin x = \sum_{n=1}^{\infty} b_n \sin nx \quad ; \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin x \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]\,dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-1)x}{(n-1)} - \frac{\sin(n+1)x}{(n+1)}\right]_0^{\pi} = \frac{1}{\pi}[0]$$

$\therefore \quad b_n = 0$

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Ex. 2. Find the Fourier series of function $f(x) = x$ in $[0, \pi]$ in terms of sine & cosine series.

Solⁿ: - The function $f(x) = x$ may be extended to an odd function $f(x)$ in $-\pi < x < \pi$ so that

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{———} \quad ⊛$$

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$; \; b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx.$

$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^\pi$

$= \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right] = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$

$\therefore \; b_n = \frac{2}{n}(-1)^{n+1} = \Big\{$

(*) $\Rightarrow$

$x = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1}\sin nx$

$\therefore \; x = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$

<u>In terms of Cosine</u> :- The function $f(x) = x$, $0 \le x < \pi$ may be extended to an even function $f(x) = |x|$ in $[-\pi, \pi]$

$\therefore \; x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx$ ——— (*)

$; \; a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^\pi = \frac{\pi^2}{\pi}.$

$\therefore \; \boxed{a_0 = \pi}$

$\because \; f(x) = |x| = \begin{cases} x & ; \; x \ge 0 \\ -x & ; \; x < 0 \end{cases}$

Also $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi \sin n\pi}{n} + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos n \cdot 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\therefore a_n = \begin{cases} 0 & ; \ n \text{ is even} \\ -\frac{4}{\pi n^2} & ; \ n \text{ is odd} \end{cases}$$

$\therefore$ (A') $\Rightarrow$ $x = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\}$

Other Form of Fourier Series

The Interval ~~[-l, l]~~ $[-l, l]$

Let $y = \frac{\pi x}{l}$ so that $-\pi \le y \le \pi$ when $-l \le x \le l$

& $f(x) = f\left(\frac{ly}{\pi}\right) = F(y)$

If 'f' satisfies the condition of Main Theorem in $[-l, l]$ then 'F' also satisfies the same condition in $[-\pi, \pi]$ / $[0, 2\pi]$

The sum of series $\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right]$ (30)

i $a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l}\, dx$.

$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\frac{n\pi x}{l}\, dx$.

is $\frac{1}{2}[f(x^+) + f(x^-)]$ $\forall$ x between $-l$ to l.

and is $\frac{1}{2}[f(l^-) + f(-l^+)]$ for $x = \pm l$

Any Interval $[a, b]$:- If f satisfies the conditions of Main Theorem in $[a, b]$, then the sum of series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi x}{b-a}\right) + b_n \sin\left(\frac{2n\pi x}{b-a}\right)\right]$$

i $a_n = \frac{2}{b-a}\int_a^b f(x) \cos\left(\frac{2n\pi x}{b-a}\right) dx$

$b_n = \frac{2}{b-a}\int_a^b f(x) \sin\left(\frac{2n\pi x}{b-a}\right) dx$. is

$\frac{1}{2}[f(x^+) + f(x^-)]$ $\forall$ x b/w a & b &

$\frac{1}{2}[f(a^+) + f(b^-)]$ at $x = a, b$ and is periodic with period $(b-a)$